How mathematics confronts its paradoxes

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Abstract. Paradoxes in mathematics show surprisingly many common features. The paper analyzes the historical development of the language of the particular mathematical theory (i.e., algebra, calculus, and predicate logic, respectively) and argues that the paradoxes occur at a particular phase of the historical development of the language. It argues that the paradoxes exhibit the expressive boundaries of the language of mathematics.

Jak se matematika vypořádává se svými paradoxy. Paradoxy v matematice vykazují překvapivě mnoho společných rysů. Tato studie analyzuje historický vývoj jazyka té které matematické teorie (algebry, infinitesimálního počtu, případně predikátové logiky) a dovozuje, že paradoxy se vyskytují v konkrétní fázi historického vývoje jazyka. Ukazuje se, že paradoxy jsou projevem expresivní hranice jazyka matematiky.

Keywords: paradox ● language of mathematics

Probably the first attempt at a systematic interpretation of the paradoxes in exact sciences was Kant's *Critique of Pure Reason* [Kant 1781]. Kant examined a range of issues, such as the question of the infinity of the extension of space and time, their divisibility etc. and showed that all efforts to answer them inevitably lead to paradoxes. According to Kant the attempts to answer these questions lead to paradoxes rising from the fact that reason is thereby exceeding the limits of its competence, which extends only as far as the limits of possible experience. He writes: „… the principles of pure understanding are only of empirical but never of transcendental use; and it follows that beyond the realm of possible experience there can be no synthetic a priori principles at all.“ [Kant 1781, p. 312].

Kant's analysis of the antinomies is significant in several respects. On the one hand Kant shows that the antinomies are not accidental mistakes or oversights but rather a *systematic phenomenon* that points toward an important feature of reason itself. „Transcendental appearance, however, does not disappear, even if it is revealed and its nothingness is perfectly recognized by means of transcendental criticism. The reason for this is that our intellect contains the basic rules and maxims of its use, which look just like objective principles, which causes that the subjective necessity of a certain union of our concepts in favor of our understanding is taken to be an objective necessity determining the things in themselves. It's an illusion, which cannot be avoided...“ [Kant 1781, s. 236].

In addition, it is interesting that Kant interprets the antinomies as boundary phenomena; as a result of *surpassing some boundaries*, in this case the boundaries of possible experience. Both of these characterizations of antinomies remain valid to this day.

Despite these undoubtedly positive insights into the nature of the paradoxes Kant's theory has also some problematic features. The first is that Kant *does not consider paradoxes in mathematics* but locates them only into physics (the antinomies of pure reason), psychology (transcendental paralogisms) and theology (ideals of pure reason). It is strange, because the work of George Berkeley *The Analyst or, a Discourse addressed to an Infidel Mathematician*, in which Berkeley points out the paradoxes in Newton's theory of fluxions and fluent, was published in 1734, i.e. almost fifty years before *Kant's Critique of Pure Reason*. Nevertheless, what Berkeley actually showed in his book is in many respects analogous to what Kant wrote about the antinomies of pure reason.

The aim of the present paper is it to show that *paradoxes occur also in mathematics*, and that the paradoxes in mathematics have both features by means of which Kant characterized the antinomies of pure reason: they are *systematic* (i.e. not mere mistakes or oversights) and they *relate to a boundary* (i.e. the same practice applied within the boundaries does not lead to paradoxes). The existence of paradoxes in mathematics casts a shadow on Kant's philosophy of mathematics and his interpretation of the antinomies as resulting from the efforts of reason to surpass the limits of possible experience. In mathematics it seems more natural to see the paradoxes not as a result of surpassing the limits of possible experience, but the boundaries of some symbolic language. I will therefore tie paradoxes to the boundaries of language and not the limits of possible experience. But despite this modification, their systematic nature and relation to a boundary remains preserved.

So we come to another peculiarity of Kant's theory, which is the idea that *the paradoxes cannot be removed*. „Therefore, the dialectical theorem of pure reason must be different from all sophistical sentences that do not concern any questions, toss with just any faith but only such, which must necessarily encounter every human mind in its thinking; and secondly, this precept with its antithesis not only creates unnatural appearances that will disappear as soon as we look into it, as well as natural and inevitable appearance that still, even though he is no longer subject, confusing, although it does not lie, and which can therefore be disposed of but can never be destroyed .... it must arise contradictions that can not be avoided, no matter what we do." [Kant 1781, s. 283 a 284]. If we interpret the paradoxes as surpassing the boundaries of a certain formal language, then a reform of that language can open up a possibility to eliminate the particular paradoxes.

History of mathematics provides a wealth of examples that show how various paradoxes appeared in mathematics, and how mathematicians created technical means of defense against them. In the first four chapters I will discuss four paradoxes: the *incommensurability* of the side and the diagonal of a square, the *casus irreducibilis* that emerged by solving cubic equations, the paradoxes of the *infinitesimal calculus* and the paradox of the *set of all sets*. In all four cases I will try to interpret the paradox as encounters of the boundaries of the particular symbolic language (of arithmetic, algebra, the calculus, or set theory respectively) and to show how mathematicians dealt with these paradoxes. (A broader historical context of these paradoxes can be found in Kvasz 2008, pp. 11–84.)

1. The boundaries of the language of arithmetic–incommensurability

For a period of several centuries, the development of mathematics was confined to practical issues of economics and trade. Therefore the main mathematical discipline was arithmetic. The mathematical texts that have survived from ancient Egypt and Mesopotamia are collections of practical problems, together with solutions. If a problem concerns geometry, the text usually lacks any visual image and in the rare cases when the image is present, it is not clear what exactly is represented. In ancient Greece for the first time we encounter geometrical texts, in which the particular geometrical objects have a well defined form and the relations among them have the character of logical necessity. The knowledge of geometry was according to tradition brought by Thales and Pythagoras from Egypt. Nevertheless, in one aspect Greek mathematics differed substantially from its predecessors in ancient Egypt and Mesopotamia; it namely contained the idea of a proof.

The paradox that I want to discuss here was discovered by the Pythagoreans. According to the Pythagorean doctrine the world is a harmony of opposites, and the essence of this harmony is expressed by numbers. The Pythagoreans associated geometry with a strange kind of "arithmetical atomism". They assumed that every line, thus also the side and diagonal of a square, are composed of a number of "units", and so the ratio of the lengths of these lines is equal to the ratio of the number of "units" that constitute them. In this way the opposites–the long and the short–are joined in a harmony, expressed in the form of a proportion of numbers. The discovery of the incommensurability of the side and the diagonal of a square, i.e. of the fact that the ratio of the lengths of these two lines cannot be expressed as a ratio of two (integer) numbers, contradicts the Pythagorean philosophy. The Pythagoreans considered incommensurability a paradox. I suggest that the incommensurability of the side and the diagonal of a square reveals the *boundaries of the language elementary arithmetic.*

The proof of the incommensurability of the side and the diagonal of a square is based on Pythagoras' Theorem. Suppose that the side of the square consists of b units and the diagonal of a units. From the Pythagorean theorem we have $a^2 = b^2 + b^2$. Thus for the numbers a and b we have

$$a^2 = 2b^2. \tag{1}$$

From this relation we see that the diagonal a is greater than the side b, and so we can write

$$a = b + c. \tag{a}$$

The number c is the difference that we get when we subtract from the diagonal a the side b. When we substitute this latter relation into the former one, we get

$$(b + c)^2 = 2b^2.$$

Using simple transformations we get

$$b^2 = 2bc + c^2. \tag{2}$$

This means that the number b^2 is greater than $2bc$, which means (after dividing the two numbers by b), that b is greater than $2c$. Therefore, we can write

$$b = 2c + d, \tag{b}$$

where d is again the difference obtained by subtracting $2c$ from b.

The reader is probably already getting bored, but now comes the point. When we substitute this expression of the number b into the equation (2) we obtain

$$(2c + d)^2 = 2(2c + d)\mathrm{c} + \mathrm{c}^2.$$

Using a simple multiplication we get

$$4c^2 + 4cd + d^2 = 4c^2 + 2cd + c^2.$$

When we subtract from both sides of the equation the equal terms, we obtain

$$c^2 = 2cd + d^2. \tag{3}$$

And we're done. The Pythagorean doctrine is refuted!!!

Why? We have to realize what we have got. The relation (3) is exactly the same as (2), that is, more precisely; the numbers b and c are in the same relation as the numbers c and d. Thus, the method of *antiphareisis*, as it was called, cannot terminate. Remember what we were doing. We have always subtracted from the longer segment (single or double) multiple of the shorter. If the original segments had a common measure s (i.e. if $a = ms$ and $b = ns$) then by repeated subtraction of the shorter segment from a longer one, as we have done it, we would have reached the stage when the *last difference* would be exactly equal to the common measure s of the original segments. But that would mean that the penultimate difference would be a multiple of this last difference s and so no new remainder would emerge, and our method would reach its end. Thus the method of *antiphareisis* is a method for finding the common measure of two segments (or the common divisor of two numbers, which was for the Pythagoreans essentially the same, because they believed that every segment is a number).

The discovery of incommensurability means that *the method of antiphareisis breaks down*. One of the pillars on which the Pythagorean theory of numbers was built, collapsed.

Within the framework of elementary arithmetic the incommensurability of the side and the diagonal of a square is a paradox that defies comprehension. The Pythagoreans called it *kai alogon kai aneidon*, ineffable and inconceivable. Only when in the 17th century mathematicians abandoned the Euclidean definition of number as a multitude of units and began to consider the real numbers, it became possible to turn the incommensurability into the positive fact that $\sqrt{2}$ is an irrational number. Here we see one of the basic strategies of overcoming paradoxes: through extension of language we turn a paradox into a positive fact. In our case, we introduce the number $\sqrt{2}$ that expresses the length of the diagonal of the unit square and the paradox is gone. We will meet this strategy in the following text several times.

The reader who knows the standard indirect proof of the incommensurability of the side and the diagonal that can be written in three lines may ask why I instead presented the proof that uses *antiphareisis*. The reason is that it is likely that this is the form in which incommensurability was originally discovered (probably instead of the square it was with the pentagon). There is also a philosophical reason, because the method of *antiphareisis* is similar to the form, in which Kant presents his antinomies. Here, just like in the case of extension or divisibility of space we are confronted with the thesis and antithesis, asserting that some process will terminate or not.

2. The boundaries of the language of algebra–the casus irreducibilis

The reconquista of Toledo in 1085 opened to the European scholars the opportunity to become acquainted with several works of Greek and Arabic mathematics from the Arabic manuscripts. In 1142 Euclid's *Elements* have been translated into Latin, and in 1145 the *Short book on algebra and al-muqábala* from Abu Abdullah Muhammad ibn Músa al-Chwárizmí. The habit to formulate the solution of a problem in the form of *verbal rules* that was so characteristic for the work of al-Chwárizmí lasted well until the 16th century. Thus also the first result of European mathematics that surpassed the knowledge of the ancient world, was written in this form. It was the solution of the cubic equation, published in 1545 in the book *Ars Magna Sive de Regulis Algebracis* by the Italian mathematician Girolamo Cardano.

Cardano formulated the cubic equation as: "*De cubo et rebus aequalibus numero*" where *cubo* is derived from *cubus*, meaning the third power of the unknown; *rebus* is derived from *res*, meaning the first power of the unknown; and *numero* is derived from *numerus*, meaning is the member. Cardano gives the solution of this equation in the form: „Cube one-third of the number of things; add to it the square of one-half of the number; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. You will then have *binomium* and its *apotome*. Then subtracting the cube root of the *apotome* from the cube root of the *binomium*, that which is left is the thing.“

Here "to duplicate" means to write two times the same next to each other and the words "*binomium*" and "*apotome*" refer to certain algebraic expressions. In order to allow the reader *to see* what Cardano *was doing*, I write down the equation using today's symbols as $x^3 + bx = c$ and its solution given by the formula:

$$x = \sqrt[3]{\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}} \; . \qquad (4)$$

In Cardan's times there were no formulas and Cardano used exclusively the verbal form. He illustrated his general rule on a particular equation $x^3 + 6x = 20$, the solution of which he writes as:

„*RV: cub: R: 108 p: 10 m: RV: cub: R: 108 m: 10*“

where *RV* is *radix universalis*; *cub* indicates that the root is of the third degree; *R* is *radix*; *p* stands for *plus*; *m* for *minus*. Cardano was not able to express the coefficients of the equation in a general form. The formula refers to a solution of a particular equation. In our symbolism it has the form:

$$\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10} \; .$$

On the one hand, the solution of the cubic equation is a landmark of the development of European mathematics. Their solution is the first result of Western mathematics that surpassed the level of the ancient heritage. Thus, the year 1545 can be seen as the first date when we knew something Euclid, Archimedes or Apollonius did not know. On the other hand, the solution of the cubic equation led to a paradox. Cardano discovered the paradox by trying to solve the equation of the type "De cubo aequali rebus et numero" what in our symbolism means $x^3 = bx + c$. The reason for the need to differentiate various types of equations was that the mathematicians of the 16th century did not know negative numbers. Cardano found for the equation $x^3 + bx = c$ a rule analogous to the previous ones. I will not present it in its original verbal form, but only its transcription into modern symbolism:

$$x = \sqrt[3]{\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 - \left(\frac{b}{3}\right)^3}} + \sqrt[3]{\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 - \left(\frac{b}{3}\right)^3}} \quad . \tag{5}$$

When we compare this formula with the one above, we find that what has changed is only that a few occurrences of + were replaced by – and vice versa. So below the square root the sum was replaced by difference. Nevertheless, this tiny change has far-reaching consequences. If we take a particular equation, for instance the $x^3 = 7x + 6$, which was considered also by Cardano, we can see, that one of its solution is $x = 3$ (the other two solutions $x = -1$ and $x = -2$ were not considered in Cardano's times). However, when we substitute the coefficients $b = 7$ and $c = 6$ into formula (5), we obtain something incomprehensible–below the sign of the square root a negative number occurs:

$$x = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}} \ .$$

Cardano called this *casus irreducibilis*, the unsolvable case. In retrospect, this is probably the first complex number in the history of mathematics. It resembles the discovery of the incommensurability of the side and the diagonal of a square. In both cases we are confronted with the limits of language. The language (of arithmetic or of algebra) is faced with a situation in which it fails. In the case of the *casus irreducibilis* some terms arise which, strictly speaking, do not make sense. And the therapy is also similar to that in the previous case—we must extend the universe of numbers. When we include complex numbers into our universe, the square root of a negative number will no longer pose any problem, just like after including the irrational numbers the $\sqrt{2}$ posed no problems. For Cardano, however, who discovered this paradox, the *casus irreducibilis* was a mysterious phenomenon, and it was necessary to wait almost two hundred years, until complex numbers were sufficiently understood so that the square roots of negative numbers were accepted as genuine quantities. (A more detailed reconstruction of the development of algebra is in Kvasz 2006.)

It seems that also this case can be brought under the umbrella of Kant's account of the paradoxes. At least in the period between 1545 (when Cardano discovered 'the first complex number') and 1799 (when Gauss introduced the complex plane) mathematicians were confronted with the irresistible temptation to work with complex numbers as if they were objects (i.e. transcendently real), even though they were not able to explain what they really are (i.e. construct them within the limits of possible experience). The appeal of Gauss' construction of the complex plane is precisely that it gave complex numbers a model within the empirical realm. So we have all the Kantian components of an antinomy–

their *systematic nature*, their relation to some *boundary* (in this case the boundary separating the positive numbers from the negative ones) and their *irresistible* nature.

3. The boundaries of the language of the calculus

The paradoxes of differential and integral calculus are in many ways analogous to the previous examples. Newton in a letter from 1714 writes: "Fluxions and moments are quantities of different kind. Fluxions are final movements, while moments are infinitely small parts. I use letters with dots to indicate fluxions and multiply the fluxion with the letter *o*, to make them infinitely small. In proofs of theorems I always write the letter *o*, and I proceed by the methods of Euclid's geometry without any approximations. In calculations I make approximations of the most varied kinds, and sometimes I omit *o*."

Newton called *fluxion* what we now call the derivative of a function, and thus it is a finite quantity that characterizes the speed with which the original quantity (that he called fluent) changes. In contrast, he called *moment* the infinitesimal increment of a quantity, and so it is infinitely small. The *moment of a quantity* (or of a function or of a fluent) y is equal to the product $\dot{y}o$ of its fluxion $\dot{y}$, which indicates how quickly the quantity y changes in time with the infinitely short moment of time o, during which the moment appeared. As an illustration, we can present Newton's calculation of the derivative of the fluent – x^3. In Newton's notation this procedure is as follows:

$$y = x^3 \tag{6}$$

$$\begin{aligned} y + \dot{y}o &= \left(x + \dot{x}o\right)^3 \\ y + \dot{y}o &= x^3 + 3x^2\dot{x}o + 3x\dot{x}\dot{x}o \;\; + \dot{x}\dot{x}\dot{x}ooo \end{aligned} \tag{7}$$

We can use (6) to get rid of the finite terms and then divide both sides by *o*. We thus obtain:

$$\dot{y} = 3x^2\dot{x} + 3x\dot{x}\dot{x}o + \dot{x}\dot{x}\dot{x}o \tag{8}$$

$$\frac{\dot{y}}{\dot{x}} = 3x^2 + 3x\dot{x}o + \dot{x}\dot{x}oo\,o \tag{9}$$

$$\frac{\dot{y}}{\dot{x}} = 3x^2 \tag{10}$$

In the last step we neglected the infinitely small quantities that contained *o*. Newton thus received the correct result; the derivative of the function $y = x^3$ is indeed $y = 3x^2$.

The calculation of the derivative using Leibniz's notation is practically the same. In a letter of 1716 Leibniz wrote: "When our friends in France disputed, I testified to them that I do not believe that there actually are some infinitely large or infinitely small quantities. The letter *d*, just like the imaginary roots in algebra, are just fiction that, nevertheless, can be used for brevity or when we want to speak generally." The above quotations indicate some doubts of the founders of the differential and integral calculus according the nature of quantities, which they used in their calculations.

George Berkeley presented a witty criticism of the differential and integral calculus in his book *The Analyst, or a discourse addressed to an Infidel Mathematician* that appeared in Dublin in 1734. In his book Berkeley makes a parody of Newton's theory of fluxions and fluents (but all his objections apply with equal force to Leibniz's theory, based on the use of differentials). Berkeley shows that the calculus is based on a logical error. Fortunately, not a single one but on a multitude of errors, which compensate each other so that the calculus eventually brings us to the correct result. Berkeley here refers to Newton's method described above, in which during the calculation a certain quantity is considered as different from zero, so that we could divide by it (in the transition from the line (7) to (8) we divided by the quantity *o*). After the successful completion of these divisions we put this quantity equal to zero (in the transition from the line (9) to (10)). Berkeley correctly objects that a variable either *is equal to zero*, but then it is equal to zero throughout the entire calculation and thus it is not possible to divide by it, or it *is not equal to zero*, but then it is not equal to zero throughout the entire calculation and so it cannot be ignored at the end. So, according to Berkeley the differential calculus is all mistaken. That it, nevertheless, leads to correct results, is possible only due to compensation of errors.

Berkeley's criticism of the differential calculus pointed to the insufficient justification of the formal methods, on which the differential calculus was built. The efforts to create foundations of the calculus span the entire 18th and 19th centuries. One of the most important attempts to build the calculus on solid foundations was presented by Augustin Cauchy in his famous *Cours de l'Analyse* from 1821. Cauchy found a way to exclude Newton's fluxions and Leibniz's differentials from the calculations and to base the entire analysis on the concept of limit. When he tried to justify the concept of limit, however, Cauchy turned to the properties of the real line. It seems that the criticism presented by Berkeley really revealed the boundaries of the language of the differential and integral calculus, because almost all attempts to answer his criticism turned to some other language.

Here we are dealing with a strategy to confront a paradox that is different from the strategy adopted in the case of the incommensurability or the *casus* *irreducibilis*. Mathematicians like Cauchy, Bolzano, or Weierstrass have not tried to enrich the language of the differential and integral calculus by some new kind of quantities (even though the strategy of extending the realm of numbers was eventually also successful, but only in 1962, when Abraham Robinson proposed his *Nonstandard analysis*). Instead of an extension of the system of numbers mathematicians *shifted the problem* of the foundations of differential and integral calculus on the shoulders of some other theory–the theory of real numbers. Around 1872 Dedekind, Weierstrass, and Cantor independently proposed a construction of real numbers and started the process of arithmetization of mathematical analysis. The constructions of Dedekind, Weierstrass, and Cantor, however, have one fundamental weakness. They assumed the existence of an actually infinite set of objects (in Dedekind's construction it was the set of all rational numbers, in the case of Cantor and Weierstrass it was the set of all sequences of rational numbers). Even if the assumption of the existence of all rational numbers seems harmless, some mathematicians considered the problem of grasping an actually infinite set of objects as not clear enough.

Therefore Dedekind (1888), Peano (1889) and Frege (1893), again independently, offered three alternative constructions of the system of all natural numbers as the canonical infinite system of objects. So around 1890 a second *shift of the problem* of foundations of the differential and integral calculus occurred. Now the foundation, on which the building of the mathematical analysis should be erected, was arithmetic of natural numbers. It seemed that this time the work started at the beginning of the century by Cauchy, finally reached its completion, and the *strategy of shifting of the problem* of foundations was crowned with success. But suddenly some logical paradoxes emerged and the entire building collapsed.

It seems that the strategy of gradual shifting of the problem of foundations is not as effective in dealing with paradoxes as the strategy of extending the language. However, by each such shifting the problem gets imbedded into a new context and so each shift allows its deeper analysis and better understanding. From the very beginning it was clear that the problem of the foundations of the calculus has something to do with the infinite, but the successive shifts of the problem clarified in what respect the concept of infinity is involved in the foundations of the calculus. As many of the Kantian antinomies were also tied to the infinity, I think it is plausible to include also this paradox among the examples of paradoxes in mathematics that illustrate Kant's notion of an antinomy.

4. Boundaries of the language of predicate calculus

One of the first logical paradoxes was discovered in 1901 by Bertrand Russell in Frege's system of the foundations of arithmetic from 1893. Frege was surprised by the appearance of a paradox in his theory and pointed out that the same paradox occurs also in Dedekind's system of the foundations of arithmetic from 1888. This observation of Frege can be supplemented by a note that the same paradox can also be derived in Peano's system [see Gillies 1982, p. 83–93]. This shows that these paradoxes are not the result of some mistakes made by the mentioned authors. Their systems are conceptually so different that the occurrence of the same paradox in all of them marks rather the logical boundaries of the language of mathematics. The problem was that all three used quantification over classes (i.e. from our point of view second-order logic). The paradoxes arise as a result of the 'careless' use of second-order quantification.

The simplest formulation of the paradox that appeared in the works of Frege, Dedekind, and Peano can be given in the language of set theory. A detailed analysis of the paradox can be found in the book [Gillies 1982], where a comparison of the formulations of the paradox in all three systems is presented. In the language of set theory this paradox is usually formulated as the paradox of the set of all sets (that do not contain themselves as element). Assume thus that M is a set consisting of all those sets that do not contain themselves as an element. Formally written

$$M = \{x;\ x \notin x\}.$$

Let us ask the question whether the very set M contains itself as an element? We do not know, so let's consider both alternatives.

Assume that M is its own element, i.e. $M \in M$. If M is an element of the set M, it must satisfy the condition that is true of all elements of the set M, namely that $x \notin x$. When we substitute M for the variable x, we get $M \notin M$. Thus M is not the element of itself, what contradicts the assumption.

Assume therefore that M is not its own element, i.e. $M \notin M$. This means that M satisfies the condition by means of which we determine whether a set belongs to M or not, namely the condition $x \notin x$. Given that M satisfies this condition, it must be included in M, because M contains all the sets that satisfy this condition, and thus we obtain that $M \in M$, what contradicts what we have assumed.

We see that whether we assume of set M that it is an element of itself or not, in both cases we obtain a contradiction. This paradox may at first glance seem an artificial trick. It may seem that the set M was created just to construct the paradox, but no one would ever dream of doing such a thing. Nevertheless, this impression is wrong. Russell found this paradox in the system of Frege, and Frege found in the system of Dedekind, and finally it turned out that it occurs also in the system of Peano. Thus it is not an artificial trick, but a consequence of the fundamental principles on which the foundations of arithmetic proposed by the three mentioned mathematicians were built.

The logical paradoxes are interesting because in their overcoming we can find a third strategy of dealing with a paradox. Besides the *strategy of language extension*, used in the case of the *incommensurability* and the *casus irreducibilis*, leading to the introduction of the irrational and the complex numbers, and the *strategy of shifting the problem* that was used in the case of the calculus, where the problems concerning the infinitely small quantities were gradually moved into arithmetic, here we are dealing with something that can be described as the *strategy of suppressing the paradox*. The first version of this strategy was presented in 1908 by Ernst Zermelo, who created the first axiomatic system of set theory. In building his system Zermelo restricted the language of set theory so that sets similar to the set *M* could not be defined in the language. It is a kind of political correctness where we believe that a problem disappears when the language does not enable its formulation.

5. Concluding remarks

As a first interesting result of our analysis we can take the discrimination of three strategies how mathematicians respond to paradoxes: the strategy of *language extension*, the strategy of *shifting the problem*, and the strategy of *suppressing the paradox*. In the last one it is not difficult to recognize Lakatos' monster barring and exception barring method. Nevertheless, it seems that the other two strategies were not analyzed by Lakatos. So it could be interesting to confront the Kantian theory of antinomies with the Lakatosian theory of *Proofs and Refutations* (for more details see Kvasz 2002).

As I said at the beginning, according to Kant there is a fundamental difference between mathematics and physics consisting in the fact that *there are no paradoxes in mathematics*, while in physics their occurrence cannot be avoided. I assume that the above text casts doubt on this view. It shows that mathematics is exposed to paradoxes to at least the same extent as physics. It is an open question whether the paradoxes that I have described in previous chapters can be interpreted as phenomena analogous to the Kantian antinomies. I believe they can. In all above mentioned paradoxes we could find those features, by means of which Kant characterized his antinomies, namely their systematic nature and that on the road towards the paradox certain rule that works well in a limited area (of possible experience), was used in an unlimited way. The tendency to use the rule for the cubic equation of the kind *Cubo aequali rebus et numero* in all cases, or to apply the rules of quantification also to classes, lies in the nature of reason itself. It is the tendency to pass from the conditional to the unconditional, on which Kant bases his analysis of the antinomies.

So it seems that there is a problem within Kant's philosophy of mathematics. I believe that if we stop separating mathematics from physics as Kant did and start to base its philosophy on language instead of perception, it will be possible to preserve several of the insights by which Kant enriched the philosophy of exact sciences. Mathematics can be thus incorporated into his analysis of antinomies of pure reason. For an orthodox Kantian such an approach is probably unacceptable, but it is questionable whether making Kant's philosophy a sterile corpse exposed in the showcases of the museum of the history of Western thought is the only way how to deal with it.

Another question of wider significance is the problem that according to Kant the antinomies belong to the very nature of reason and thus they *cannot be removed*. As shown by the development of physics the antinomies can indeed be removed. Let us take for example the antinomy of *finiteness versus infinity of space*. General theory of relativity eliminated this antinomy when it replaced the Euclidean space of Newtonian physics by the curved space-time. For general theory of relativity Kant's antinomy does not work; the question whether space is finite or infinite is not a speculative problem. It is an empirical question that depends on the distribution of matter in the universe.

Nevertheless, Kant's antinomies do not lose their significance. They should be qualified with respect to the language of the theory in which the antinomy is formulated. Thus I suggest interpreting the antinomy of finiteness versus infinity of space as pointing to the *external character of space in Newtonian physics* (that was first noticed by Mach). This antinomy is thus not a characteristic feature of human reason, but of language. And not of language as such, but of the language of Newtonian mechanics. The antinomy of finiteness versus infinity of space can be therefore interpreted as an indication of *the boundaries of the language of Newtonian physics* (for more details see Kvasz 2013). This is actually the form in which I described all the paradoxes in mathematics. And in this form Kant's antinomy of finiteness versus infinity of space retains its validity and tells something fundamental about the nature of Newtonian physics.

Of course, for Kant Newtonian physics was the only physics that he knew, and so the boundaries of its language appeared to him as limits of physics as such, and actually as limits of the human reason. As we today have a number of fundamental physical theories (such as the theory of relativity or quantum mechanics), it is easier for us to see that the antinomies studied by Kant are in fact related to one specific theory, namely Newtonian mechanics and thus they have nothing to do with reason as such. And physics faces these antinomies in very the way as mathematics (described in the previous chapters), namely with a change of language.

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Summary

Paradoxes in mathematics such as the *casus irreducibilis* [Cardano 1545], the paradoxes of the calculus [Berkeley 1734] or Russell's paradox [Russell 1903] show surprisingly many common features. It is possible to see these paradoxes as linguistic phenomena occurring at a specific stage in the development of the particular theory. It seems that even though each paradox taken in isolation is well understood, the paradoxes as a general phenomenon still lack sufficient historical analysis. The paper analyzes the historical development of the language of the particular mathematical theory (i.e., algebra, calculus, and predicate logic, respectively) and argues that the paradoxes occur at a particular phase of the historical development of the language; it characterizes that stage as the stage when in the language we begin to construct representations of representations. It argues that the paradoxes exhibit the expressive boundaries of the language of mathematics as introduced in [Kvasz 2008]. That is why these paradoxes exhibit several common features–they correspond to the same epistemological phenomenon, namely expressive boundaries of language.

Resumé

Paradoxy v matematice jako *casus irreducibilis* [Cardano 1545], paradoxy infinitesimálního počtu [Berkeley 1734] nebo Russellův paradox [Russell 1903] vykazují překvapivě mnoho společných rysů. Je možné na ně nahlížet jako na lingvistické fenomény vyskytující se v určitém stavu vývoje konkrétní teorie. I když je každý výše uvedený paradox sám o sobě dobře prostudován, paradoxy jako obecný fenomén stále postrádají dostatečnou historickou analýzu. Tato studie analyzuje historický vývoj jazyka té které matematické teorie (algebry, infinitesimálního počtu, případně predikátové logiky) a dovozuje, že paradoxy se vyskytují v konkrétní fázi historického vývoje jazyka; charakterizuje tuto fázi jako takovou, v níž v jazyce začínáme vytvářet reprezentace reprezentací. Ukazuje se, že paradoxy jsou projevem expresivní hranice jazyka matematiky, zavedené v [Kvasz 2008]. Proto paradoxy vykazují společné rysy – korespondují s týmž epistemologickým fenoménem, především s expresivními hranicemi jazyka.

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